

Numbers Divisible by 3

Any number is divisible by 3 if the sum of the digits is divisible by 3. For example 123 is $1+2+3 = 9$ so it is divisible by 3.

Proof:

Two Digit Proof:

$$n = 10a + b \quad \text{Let } n \text{ be the number, } a \text{ and } b \text{ the digits}$$

$$a + b = c \quad \text{The sum of the digits is } c$$

$$b = c - a \quad \text{Re-arrange the sum of the digits}$$

$$n = 10a + c - a \quad \text{Substitute into the first equation}$$

$$n = 9a + c \quad \text{Simplify}$$

Given $a + b \% 3 = 0$, or the sum of the digits is divisible by 3, c is divisible by 3

Both $9a$ and c are divisible by 3, allowing a 3 to be factored out, guaranteeing $n \% 3 = 0$

Three Digit Proof:

$$n = 100a + 10b + c \quad \text{Let } n \text{ be the number, } a, b, \text{ and } c \text{ the digits}$$

$$a + b + c = d \quad \text{The sum of the digits is } d$$

$$c = d - b - a \quad \text{Re-arrange the sum of the digits}$$

$$n = 100a + 10b + d - b - a \quad \text{Substitute into the first equation}$$

$$n = 99a + 9b + d \quad \text{Simplify}$$

Given $a + b + c \% 3 = 0$, or the sum of the digits is divisible by 3, d is divisible by 3

$99a$, $9b$, and d are divisible by 3, allowing a 3 to be factored out, guaranteeing $n \% 3 = 0$

X Digit Proof:

$digit_0$ is the least significant digit, $digit_x$ is the most significant digit

$$n = \sum_{i=0}^x digit_i 10^i \quad \text{Let } n \text{ be the number and } digit_i \text{ the digits}$$

$$s = \sum_{i=0}^x digit_i \quad \text{The sum of the digits is } s$$

$$digit_0 = s - \sum_{i=1}^x digit_i \quad \text{Re-arrange the sum of the digits}$$

$$n = \sum_{i=1}^x digit_i 10^i + digit_0 10^0 \quad \text{Re-write the first equation}$$

$$n = \sum_{i=1}^x digit_i 10^i + digit_0 \quad \text{Simplify}$$

$$n = \sum_{i=1}^x digit_i 10^i + s - \sum_{i=1}^x digit_i \quad \text{Substitute the re-arranged sum of the digits}$$

$$n = \sum_{i=1}^x digit_i 10^i - digit_i + s \quad \text{Combine the summations}$$

$$n = \sum_{i=1}^x \text{digit}_i 10^i - 1 + s \quad \text{Factor}$$

Need to prove $\forall i \quad 10^i - 1 \pmod{3} = 0$

$$10^i - 1 = 10^{i-1+1} - 1 \quad \text{Add and subtract 1 from the exponent}$$

$$10^{i-1+1} - 1 = \frac{10^{i-1+1} - 1}{9} \cdot 9 \quad \text{Multiply and divide by 9}$$

$$\frac{10^{i-1+1} - 1}{9} \cdot 9 = 9 \frac{10^{i-1+1} - 1}{10 - 1} \quad \text{Re-write the bottom 9 as } (10 - 1)$$

$$9 \frac{10^{i-1+1} - 1}{10 - 1} = 9 \sum_{z=0}^{i-1} 10^z \quad \text{Re-write equation as a summation}$$

$$n = \sum_{i=1}^x \left(\text{digit}_i 9 \sum_{z=0}^{i-1} 10^z \right) + s \quad \text{Replace original equation with summation}$$

$$n = 9 \sum_{i=1}^x \left(\text{digit}_i \sum_{z=0}^{i-1} 10^z \right) + s \quad \text{Pull the 9 out of the summation}$$

9 and s are divisible by 3, allowing a 3 to be factored out, guaranteeing $n \pmod{3} = 0$